Note :-

When $n \to \infty$ $\beta_1 \to 0$

$$\beta_2 = \lim_{n\to\infty} 3\left[\frac{n-2}{n-4}\right]$$

$$= 3 \lim_{n\to\infty} \left[\left\{\frac{1-2/n}{1-4/n}\right\}\right] \to 1$$

curtosis

$$= 3$$

∴ Hence for large degrees of freedom 't' distribution tends to normal distribution.

Applications of t Distribution :-

1. To test if the Sample mean ($\bar{x}$) differs significantly from the hypothetical value μ of the population mean.

2. To test the significance of difference between two Sample means.

3. To test the significance of an observed sample Correlation Co-efficient and sample regression Co-efficient.

4. To test the significance of observed Partial Correlation Co-efficient.

(119)

$$= \frac{n^r}{\beta(1/2, n/2)} \beta\left(\frac{n}{2} - r, r + \frac{1}{2}\right)$$

$$= \frac{n^r \sqrt{1/2 + n/2}}{\sqrt{1/2}\sqrt{n/2}} \cdot \frac{\sqrt{n/2 - r}\sqrt{r + 1/2}}{\sqrt{n/2 - \cancel{r} + \cancel{r} + 1/2}}$$

(2)

$$= \frac{n^r \sqrt{\frac{n+1}{2}}}{\sqrt{1/2}\sqrt{n/2}} \times \frac{\sqrt{n/2 - r}\sqrt{r + 1/2}}{\sqrt{n/2 - \cancel{r} + \cancel{r} + 1/2}}$$

(1)

$$= \frac{n^r \sqrt{n/2 - r}\,(r - 1/2)(r - 3/2)}{\sqrt{1/2}\,(n/2 - 1)(n/2 - 2)}$$

$3/2 \quad 1/2 \quad \sqrt{1/2}$

$(n - 2r) \quad n/2$

$$= \frac{n^r (2r - 1)(2r - 3)}{(n - 2)(n - 4)} \qquad 3 \cdot 1$$

$(n - 2r)$

$$\mu_{2r} = n \frac{1}{n - 2} = \boxed{\frac{n}{n - 2}}$$

put $r = 1$

$$\mu_4 = \frac{n^2 (3)(1)}{(n - 2)(n - 4)} = \frac{3n^2}{(n - 2)(n - 4)}$$

put $r = 2$

$n^2 (2(2) - 1)$

$(2(2) - 3)$

$= n^2 (3)(1)$

$= 3n^2$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3n^2 / (n - 2)(n - 4)}{(n / n - 2)^2}$$

$\frac{3n^2}{(n-2)(n-4)}$

$\frac{3n^2}{(n-2)(n-4)}$

$$= \frac{3n^2 (n - 2)^2}{n^2 (n - 2)(n - 4)}$$

$\frac{n-2}{n-4}$

$$= 3\left(\frac{n - 2}{n - 4}\right)$$

(18)

$$= \int_{-\infty}^{\infty} t^{2r} f(t)\, dt = 2\int_{0}^{\infty} t^{2r} f(t)\, dt$$

$$= 2\,\frac{1}{B(1/2, n/2)\sqrt{n}} \int_0^{\infty} \frac{t^{2r}}{(1+t^2/n)^{n+1/2}}\, dt$$

Let $1 + t^2/n = 1/y$

$t^2/n = 1/y - 1 = 1-y/y$

$t^2 = \frac{n(1-y)}{y}$

$\frac{n-ny}{y} = n/y - n$

$2t\,dt = \frac{-n}{y^2}\, dy$

$dt = -\frac{n}{y^2 dt} dy$

when $t = 0$ $\quad y = 1$

$t = \infty$ $\quad y = 0$

$$\therefore \mu_{2r} = \frac{-2\sqrt{n}}{\sqrt{n}\,B(1/2, n/2)} \int_1^0 \frac{t^{2r}}{(1/y)^{n+1/2}} \cdot \frac{-n}{2t y^2}\, dy$$

$$= \frac{-\cancel{2}\sqrt{n}}{\cancel{2}\sqrt{n}\,B(1/2, n/2)} \int_1^0 t^{2r} (t^2)^{-1/2}\, y^{n+1/2}\, y^{-2}\, dy$$

$$= \frac{\sqrt{n}}{B(1/2, n/2)} \int_1^0 t^{2(r-1/2)} \cdot y^{n+1/2-2}\, dy$$

$$= \frac{\sqrt{n}}{B(1/2, n/2)} \int_0^1 \left[n\left(\frac{1-y}{y}\right)\right]^{r-1/2} y^{\frac{n+1}{2}-2}\, dy$$

$$= \frac{\sqrt{n}}{B(1/2, n/2)} \int_0^1 n^{r-1/2} (1-y)^{r-1/2} y^{-r+1/2}\, y^{\frac{n+1}{2}-2}\, dy$$

$$= \frac{\sqrt{n}\, n^{r-1/2}}{B(1/2, n/2)} \int_0^1 y^{-r+1/2+n/2+1/2-2}\, (1-y)^{r-1/2+1}\, dy$$

$$= \frac{n^r}{B(1/2, n/2)} \int_0^1 y^{n/2-r-1}\, (1-y)^{r+1/2-1}\, dy$$

If $\dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$ ——— (1)

and $\chi^2 = \dfrac{ns^2}{\sigma^2} = \dfrac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sigma^2}$ ——— (2)

is independently distributed as chi-square variate with (n-1) d.f. Hence Fisher's 't' is given by

$$t = \frac{\xi}{\sqrt{\chi^2/n-1}} = \frac{\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma}}{\frac{\sigma}{\sqrt{\frac{1}{n-1}\sum(x_i-\bar{x})^2}}}$$

$$= \frac{(\bar{x}-\mu)}{S/\sqrt{n}} \quad \text{——— (3)}$$

and it follows students 't' distribution with (n-1) degrees of freedom (3) is same as Students 't' defined as before. Hence Students 't' is a particular case of Fisher's 't'.

Unit-5 2(6) mar

Constants of 't' distribution:

Since f(t) is symmetrical about the line t=0, all the moments of odd order about origin vanishes,

i.e μ'_{2r+1} (about origin) = 0

In particular, r = 0, 1, 2

μ_1' (about origin) = 0 = Mean

Hence central moments coincide with moments about origin

$\therefore \mu_{2r+1} = 0$.

The moments of even order are given by $\mu_{2r} = \mu'_{2r}$ (about

$$dG(t) = \frac{1}{\sqrt{2\pi}\, 2^{n/2} \Gamma(n/2) \sqrt{n}} \left[\int_0^\infty e^{-u \frac{1}{2}(1+t^2/n)} u^{\frac{n+1}{2}-1} \, du \, dt \right]$$

using the result $\int_0^\infty e^{-ax} x^{\lambda-1} dx = \frac{\Gamma(\lambda)}{a^\lambda}$

$$dG_1(t) = \frac{1}{\sqrt{2\pi}\, 2^{n/2} \Gamma(n/2) \sqrt{n}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\left[\frac{1}{2}\left(1+\frac{t^2}{n}\right)\right]^{\frac{n+1}{2}}} dt$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{2}\sqrt{\pi}\, 2^{n/2} \Gamma(n/2) \sqrt{n} \left(\frac{1}{2}\right)^{n+1/2} \left(1+t^2/n\right)^{n+1/2}} dt$$

$$= \frac{\Gamma\left(\frac{(n+1)}{2}\right)}{2^{1/2}\, \Gamma(1/2)\, 2^{n/2} \Gamma(n/2) \sqrt{n} \cdot 2^{-\left(\frac{n+1}{2}\right)} \left(1+t^2/n\right)^{\frac{n+1}{2}}} dt$$

Om muruga

Om muruga

$$= \frac{\Gamma\left(\frac{(n+1)}{2}\right)}{\sqrt{n} \cdot \Gamma(1/2)\, \Gamma(n/2)} \cdot \frac{1}{\left(1+t^2/n\right)^{\frac{n+1}{2}}} dt \quad -\infty \le t \le \infty$$

$$= \frac{1}{\sqrt{n}\, \beta(1/2, n/2)} \cdot \frac{1}{\left(1+t^2/n\right)^{\frac{n+1}{2}}} dt$$

Which is same as the probability function of standard 't' distribution with n degrees of freedom.

Remarks:

(i) In Fisher's 't' the degrees of freedom is same as the degrees of freedom of chi-square variate.

Students 't' may be regarded as a particular case of Fishers 't'

Since $\bar{x} \sim N(\mu, \sigma^2/n)$

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Let $t = \dfrac{\xi}{\sqrt{\chi^2/n}}$ and $u = \chi^2$

$$\xi = t\sqrt{\chi^2/n}$$

$$\xi = t\sqrt{u/n}$$

$$\frac{d\xi}{dt} = \sqrt{u/n} = \frac{u^{1/2}}{n^{1/2}}$$

u diff

$$\frac{d\xi}{du} = \frac{t}{\sqrt{n}}\,\frac{1}{2}\,u^{-1/2} = \frac{t}{2\sqrt{nu}}$$

Jacobian Transformation J is

$$J = \frac{\partial(\xi, \chi^2)}{\partial(t, u)} = \begin{vmatrix} \frac{\partial \xi}{\partial t} & \frac{\partial \xi}{\partial u} \\ \frac{\partial \chi^2}{\partial t} & \frac{\partial \chi^2}{\partial u} \end{vmatrix}$$

$$= \begin{vmatrix} \sqrt{u/n} & t/2\sqrt{un} \\ 0 & 1 \end{vmatrix} =$$

Now the joint distribution of t & u becomes,

$$dG(t,u) = \frac{1}{\sqrt{2\pi}\, 2^{n/2}\,\Gamma(n/2)}\, e^{-\frac{t^2u}{2n}}\, e^{-u/2}\, u^{n/2-1} \sqrt{u/n}\; dt\, du$$

$$-\infty \le t \le \infty$$

$$0 \le u \le \infty$$

$$= \frac{1}{\sqrt{2\pi}\, 2^{n/2}\,\Gamma(n/2)\sqrt{n}}\, e^{-\frac{u}{2}\left(1+\frac{t^2}{n}\right)}\, u^{n/2-1/2}\, dt\, du$$

Integrating out u over the range 0 to ∞ marginal distribution of 't' becomes

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$$dF(t) = \frac{1}{\sqrt{\nu}\, \beta(1/2, \nu/2)(1+t^2/\nu)^{\nu+1/2}} dt$$

The factor 2 disappearing since the integral from $-\infty$ to ∞ must be unity. This is the required probability function of students 't' distribution with $\nu = (n-1)$ degrees of freedom.

Fisher's 't' defenition:

It is the ratio of standard rand normal variate to the square root of an independent chi-square variate devided by its degrees of freedom. If ξ is a $N(0,1)$ and χ^2 is an independent chi-square variate with n degrees of freedom then fisher's 't' distribution is given by

$t = \frac{\xi}{\sqrt{\chi^2/n}}$ and it follows student 't' distribution with 'n' degrees of freedom.

Derivation

Distribution of Fisher's 't'

Since ξ and χ^2 are independent, their joint probability differential is given by

$$dF(\xi, \chi^2) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} \frac{e^{-\chi^2/2} (\chi^2)^{\nu/2-1}}{2^{\nu/2}\, \Gamma\nu/2}\, d\xi\, d\chi^2$$

$-\infty \le \xi \le \infty$

$0 \le \chi^2 \le \infty$

Since x_i $(i = 1, 2 \dots n)$ is a random sample from the normal population with mean μ and variance σ^2,

$\bar{x} \sim N(\mu, \sigma^2/n)$

$\dfrac{(\bar{x}-\mu)^2}{\sigma^2/n}$, being the square of standard normal variate is a chi square variate with one degrees of freedom.

$z = \dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}}$

Also $\dfrac{ns^2}{\sigma^2}$ is a variate with $(n-1)$ degrees of freedom.

Since $\bar{x}$ and s^2 are independently distributed, $\dfrac{t^2}{n-1}$ being the ratio of two independent χ^2 variate with 1 and $(n-1)$ degrees of freedom respectively, it is $\beta_2\left(\frac{1}{2}, \frac{n-1}{2}\right)$ variate & its distribution is given by

$$dF(t) = \frac{1}{\beta(1/2, \nu/2)} \frac{(t^2/\nu)^{1/2-1}}{(1+t^2/\nu)^{\nu+1/2}} \cdot d(t^2/\nu)$$

$0 \le t^2 \le \infty$

$\nu = n-1$

$$= \frac{(t^2/\nu)^{-1/2}}{\beta(1/2, \nu/2)(1+t^2/\nu)^{\nu+1/2}} \, d(t^2/\nu)$$

$$= \frac{1/\nu \cdot 2t\,dt}{(t^2/\nu)^{1/2}\,\beta(1/2, \nu/2)(1+t^2/\nu)^{\nu+1/2}}$$

$$= \frac{2t\,dt}{\nu(t/\sqrt{\nu})\,\beta(1/2, \nu/2)(1+t^2/\nu)^{\nu+1/2}}$$

is the sample mean and $S^2 = \frac{1}{n-1}\Sigma(x_i-\bar{x})$ is an unbiased estimate of the population variance σ^2 and it follows Students 't' distribution with $\boxed{\nu = (n-1)}$ degrees of freedom with probability distribution.

$$f(t) = \frac{1}{\sqrt{\nu}\,\beta(\frac{1}{2},\frac{\nu}{2})} \cdot \frac{1}{(1+t^2/\nu)^{\nu+1/2}}$$

Derivation of Students 't' distribution

$$t = \frac{\bar{x}-\mu}{S/\sqrt{n}}$$

$$= \frac{(\bar{x}-\mu)\sqrt{n}}{S}$$

Square on both S Sides.

$$t^2 = \frac{(\bar{x}-\mu)^2 \cdot n}{S^2}$$

Since,

$$S^2 = \frac{ns^2}{n-1},$$

$$t^2 = \frac{(\bar{x}-\mu)^2 \cdot n}{ns^2/n-1}$$

$$t^2 = \frac{n(\bar{x}-\mu)^2(n-1)}{ns^2}$$

$$s^2 = \frac{1}{n}\Sigma(x_i-\bar{x})^2$$

$$ns^2 = \Sigma(x_i-\bar{x})^2$$

$$S^2 = \frac{1}{n-1}\Sigma(x_i-\bar{x})^2$$

$$(n-1)S^2 = \Sigma(x_i-\bar{x})^2$$

$$(n-1)S^2 = ns^2$$

$$S^2 = \frac{n}{n-1}s^2$$

Multiplying and dividing by σ^2,

$$\frac{t^2}{n-1} = \frac{n(\bar{x}-\mu)^2 \cdot \sigma^2}{ns^2 \cdot \sigma^2}$$

$$(x-\mu)^2/\sigma^2/n$$

January 9

Sampling Distribution

(The Number of possible Samples of size n that can be drawn from finite population of size N is NC_n.
For each of these samples we can compute a statistic say 't' ....
eg:- mean, variance etc (..) which will obviously vary from sample to sample.
The aggregrate of the various values of the statistic under consideration so obtained may be grouped into a frequency distribution which is known as the sampling distribution of the statistic)

Important day
January - 9

Standard Error:-
The Standard deviation of the sampling distribution of a statistic is known as standard error. For eg:

Eg:- $\bar{x} = \sigma/\sqrt{n}$

observed sample proportion } $'p' = \sqrt{PQ/n}$

Students 't' distribution:

(Let x_i $(i = 1, 2 \ldots n)$ be a random sample of size 'n' from a normal population with mean μ and variance σ^2.
Then Students 't' is defined by the statistic, $t = \dfrac{\bar{x} - \mu}{S/\sqrt{n}}$ wher $\bar{x} = \dfrac{1}{n} \sum x_i$

$$= \frac{e^{\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}(t_1^2\sigma_1^2 + t^2\sigma_2^2 + 2\rho t_1 t_2 \sigma_1\sigma_2)}}{2\pi} \int_{-\infty}^{\infty} e^{-w^2/2}\, dw \cdot \int_{-\infty}^{\infty} e^{-z^2/2}\, dz$$

$$= \frac{e^{\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}(t_1^2\sigma_1^2 + t^2\sigma_2^2 + 2\rho t_1 t_2 \sigma_1\sigma_2)}}{2\pi} \sqrt{2\pi}\,\sqrt{2\pi}$$

$$= e^{\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}(t_1^2\sigma_1^2 + t^2\sigma_2^2 + 2\rho t_1 t_2 \sigma_1\sigma_2)}$$

In particular if $(X,Y) \sim BVN(0,0,1,1,\rho)$

$$M_{XY}(t) = e^{\frac{1}{2}(t_1^2 + t_2^2 + 2\rho t_1 t_2)}$$

$\mu_1 + u \cdot \sigma_1 = x \qquad \Rightarrow \qquad \mu_2 + v \cdot \sigma_2 = y$

[redacted]

$$\frac{\partial x}{\partial u} = \sigma_1 \qquad \frac{\partial y}{\partial v} = \sigma_2$$

$|J|$ Jacobian Transformation

$$\frac{\partial x}{\partial v} = 0 \qquad \frac{\partial y}{\partial u} = 0$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{vmatrix} = |\sigma_1 \sigma_2| = \sigma_1 \sigma_2$$

Now $$M_{xy}(t_1,t_2) = \int_u \int_v e^{t_1(\mu_1 + \sigma_1 u) + t_2(\mu_2 + v\sigma_2)} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\{u^2 - 2\rho uv + v^2\}} |J| \, du\, dv$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_u \int_v e^{\mu_1 t_1 + t_1\sigma_1 u + \mu_2 t_2 + t_2\sigma_2 v - \frac{1}{2(1-\rho^2)}\{u^2 - 2\rho uv + v^2\}} \sigma_1\sigma_2 \, du\, dv$$

$$= \frac{e^{(t_1\mu_1 + t_2\mu_2)}}{2\pi\sqrt{1-\rho^2}} \int_u \int_v e^{-\frac{1}{2(1-\rho^2)}\left[\{u^2 - 2\rho uv + v^2\} - 2(1-\rho^2)\{t_1\sigma_1 u + t_2\sigma_2 v\}\right]} du\, dv$$

$\{v - \rho t_1\sigma_1 - t_2\sigma_2\}^2 - t_1^2\sigma_1^2 - t_2^2\sigma_2^2 - 2 \ldots$

calculation

$$u - \rho v - (1-\rho^2) t_1 \sigma_1 = w\sqrt{(1-\rho^2)}$$

$$v - \rho t_1 \sigma_1 - t_2 \sigma_2 = z$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{(1-\rho^2)}} \cdot e^{-\frac{1}{2(1-\rho^2)\sigma_2^2}\left[\left\{(y-\mu_2)-\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right\}^2\right]}$$

Thus the conditional distribution of Y for a fixed x is

$$\left(Y/_{X=x}\right) \sim N\left[\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1),\ \sigma_2^2(1-\rho^2)\right]$$

Imp 3
5 mark

M.G.F of Bivariate Normal Distribution

(30)

Let $(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

By definition $M_{XY}(t_1,t_2) = E\left[e^{t_1X+t_2Y}\right]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(t_1x+t_2y)} \cdot f(x,y)\,dx\,dy$$

$$M_{XY}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(t_1x+t_2y)} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left\{\frac{(x-\mu_1)^2}{\sigma_1^2}-2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\frac{(y-\mu_2)^2}{\sigma_2^2}\right\}}dx\,dy$$

Put $u = \frac{x-\mu_1}{\sigma_1}$ and $v = \frac{y-\mu_2}{\sigma_2}$

$u\cdot\sigma_1 = x-\mu_1$ $\Rightarrow$ $v\cdot\sigma_2 = y-\mu_2$

$\mu_1+u\cdot\sigma_1 = x$ $\boxed{\mu_2+v\cdot\sigma_2 = y}$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1\sqrt{1-\rho^2}} \cdot e^{-\frac{1}{2(1-\rho^2)\sigma_1^2}\left[\left\{x-\mu_1-\rho\cdot\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right\}^2\right]}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1\sqrt{1-\rho^2}} \cdot e^{-\frac{1}{2(1-\rho^2)\sigma_1^2}\left[\left\{x-\left(\mu_1+\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right)\right\}^2\right]}$$

Which is the probability function of a univariate normal distribution with mean and variance given by

$$E\left[X/_{Y=y}\right] = \underbrace{\mu_1+\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)}_{\text{Mean}} \text{ and } Var\left[X/_{Y=y}\right] = \underbrace{\sigma_1^2(1-\rho^2)}_{\text{Variance}}$$

Hence the conditional distribution X for a fixed y is

$$\left[X/_{Y=y}\right] \sim N\left[\mu_1+\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2),\ \sigma_1^2(1-\rho^2)\right]$$

Similarly the conditional distribution of Y for a fixed x is

$$f(Y/x) = \frac{f(x,y)}{f(x)}$$

Conditional Distribution of x for a fixed y is

$$f(x/y) = \frac{f(x,y)}{f(y)}$$

$$f(x/y) = \frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma}\right)^2 - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right\}}}{\frac{1}{\sigma_2\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}\right)}$$

$$= \frac{\cancel{\sigma_2}\sqrt{\cancel{2}\pi}}{\cancel{2\pi}\sigma_1\cancel{\sigma_2}\sqrt{1-\rho^2}} \cdot e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right] + \frac{e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}}{e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1\sqrt{1-\rho^2}} \, e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right]} \cdot e^{-\frac{1}{2(1-\rho^2)}\left(\frac{y-\mu_2}{\sigma_2}\right)^2\left[1-(1-\rho^2)\right]}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1\sqrt{1-\rho^2}} \cdot e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\left\{1-(1-\rho^2)\right\}\right]}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1\sqrt{1-\rho^2}} \, e^{-\frac{1}{2(1-\rho^2)\sigma_1^2}\left\{(x-\mu_1)^2 - 2\rho\cdot\frac{\sigma_1}{\sigma_2}(x-\mu_1)(y-\mu_2) + \frac{\sigma_1^2}{\sigma_2^2}\rho^2(y-\mu_2)^2\right\}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1\sqrt{1-\rho^2}} \cdot e^{-\frac{1}{2(1-\rho^2)\sigma_1^2}\left[\left\{(x-\mu_1) - \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right\}^2\right]}$$

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$$= \frac{1}{2\pi\sigma_1\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2(1-\rho^2)\right\}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\left\{u-\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right\}^2\right]}$$

$$= \frac{1}{2\pi\sigma_1} e^{-1/2\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left\{u-\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right\}^2}$$

Put $\frac{1}{\sqrt{1-\rho^2}}\left[u-\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right] = t$

S.N

$$\Rightarrow u-\rho\left(\frac{x-\mu_1}{\sigma_1}\right) = t\sqrt{1-\rho^2}$$

$$du = dt\sqrt{1-\rho^2}$$

$$\therefore f(x) = \frac{1}{2\pi\sigma_1} e^{-1/2\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{1-\rho^2}} e^{-\left(\frac{t^2}{2}\right)} \sqrt{1-\rho^2}\,dt$$

$$\int_{-\infty}^{\infty} e^{-t^2/2}\,dt = \sqrt{2\pi}$$

$$\therefore f(x) = \frac{1}{\sqrt{2\pi}\,\sqrt{2\pi}\,\sigma_1}\left[e^{-1/2\left(\frac{x-\mu_1}{\sigma_1}\right)^2}\right]\cdot\cancel{\sqrt{2\pi}}$$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-1/2\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \to N.D$$

Similarly, Marginal Distribution of Y.

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

i.e. $f(y) = \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-1/2\left(\frac{y-\mu_2}{\sigma_2}\right)^2}$

Similarly, Marginal distribution

UNIT - III

Bivariate Normal Distributions :

2m Let X and y be two normally correlated variables with correlation coefficient ρ and $E(X)=\mu_1$, $V(X)=\sigma_1^2$; $E(Y)=\mu_2$, $Var(Y)=\sigma_2^2$

i.e.,

Assumptions for deriving Bivariate N

(i) The Regression of Y on X is linear

(ii) The arrays are homoscedastic

i.e., variance in each array is same

The common variance of estimate of Y in each array is then given by $\sigma_2^2(1-\rho^2)$,

ρ is the correlation coefficient of X and it is independent of X.

(iii) The distribution of Y in different array is normal.

The density function of bivariate N

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left\{\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right.\right.}$$

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\cdots \frac{x-\mu_1}{}}$$

$-\infty < x$

$-\infty < y <$

Hence $\mu_1, \mu_2, \sigma_1 (>0)$,

$\sigma_2 (>0)$ and $\rho\ (-1<\rho<1)$ are the five parameters of the distributions.

(01)

Marginal Distributions of Bivariate N.D

$$e^{\frac{1}{2(1-\rho^2)}\frac{(x-\mu_1)^2}{\sigma_1^2}} - e^{\frac{2\rho u}{(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)} + e^{\frac{u^2}{2(1-\rho^2)}}$$

Marginal distribution

The Marginal distribution of X is

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$\therefore f(x) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]}$$

Put $u = \frac{y-\mu_2}{\sigma_2} \Rightarrow u\cdot\sigma_2 = y-\mu_2$

$\therefore \sigma_2\,du = dy$

$$\therefore f(x) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\left\{e^{-\frac{1}{2(1-\rho^2)}\left\{\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho u\left(\frac{x-\mu_1}{\sigma_1}\right) + u^2\right\}}\cdot\cancel{\sigma_2}\,du\right.$$

$$= \frac{1}{2\pi\sigma_1\sqrt{(1-\rho^2)}}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} e^{-\frac{1}{2(1-\rho^2)}\left\{u^2 - 2\rho u\left(\frac{x-\mu_1}{\sigma_1}\right)\right\}}du$$

Add & Subtract $\rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2$

$$= \frac{1}{2\pi\sigma_1\sqrt{(1-\rho^2)}}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma}\right)^2} e^{-\frac{1}{2(1-\rho^2)}\left\{u^2 - 2\rho u\left(\frac{x-\mu_1}{\sigma_1}\right) + \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right\}}du$$

$$= \frac{1}{2\pi\sigma_1\sqrt{(1-\rho^2)}}\, e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right.}\int_{-\infty}^{\infty} e^{\frac{-1}{2(1-\rho^2)}\left[\left\{u-\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right\}^2\right]}$$